International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Orthocomplemented Complete Lattices and Graphs

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Received October 12, 1994

The problem I consider originates from Dörfler, who found a construction to assign an orthocomplemented lattice $H(G)$ to a graph G. By Dörfler it is known that for every finite orthocomplemented lattice L there exists a graph G such that $H(G) = L$. Unfortunately, we can find more than one graph G with this property, i.e., orthocomplemented lattices which belong to different graphs can be isomorphic. I show some conditions under which two graphs have the same orthocomplemented lattice.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. First we define an operator R on the set of subsets of V as follows:

$$A \subset V, A \neq \emptyset: \quad R(A) = \{v \in V: [v, w] \in E \ \forall w \in A\}$$

$$R(V) = \emptyset$$

$$R(\emptyset) = V$$

This means $R(A)$ is the set of vertices of V which are adjacent to all vertices of A.

Further we consider the operator H defined as

$$H := R^2(A) = R(R(A))$$

H is a closure-operator and we say that a subset $A \subseteq V$ is H-closed iff $H(A) = A$. The set of all H-closed subsets of V is denoted by $H(G)$.

Now we know the following:

Theorem 1. $H(G)$ ordered by inclusion is an orthocomplemented complete lattice with $R(A)$ as the orthocomplement of an H-closed set A. The

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0020-7748/95/0800-1647$07.50/0

lattice operations $\wedge$ and $\vee$ are defined by

$$A \wedge B = A \cap B \qquad \text{for} \quad A, B \in H(G)$$

and

$$A \vee B = H(A \cup B) \qquad \text{for} \quad A, B \in H(G)$$

By Dörfler (1976) the following is known:

Theorem 2. For every finite orthocomplemented lattice L there exists a graph G such that $H(G) = L$.

This means we can represent every orthocomplemented lattice L by a graph $G = (V, E)$. But this graph is not uniquely determined.

Problem. $H(G_1)$ and $H(G_2)$ can be isomorphic for different graphs G_1 and G_2.

For example the orthocomplemented lattices which belong to the graphs shown in Fig. 1 are isomorphic.

I will show now some conditions under which two graphs G_1 and G_2 have the same orthocomplemented lattice $L \cong H(G_1) \cong H(G_2)$.

2. RESULTS OF DÖRFLER

First I note two results already known by Dörfler (1976).

For the first result we have to define an equivalence relation T on the vertex set V as follows:

Definition 1. Let $G = (V, E)$ be a simple graph. Then the equivalence relation T on the vertex set V is defined by vTw iff $[v, w] \notin E$ and every $z \neq v, w$ which is adjacent to one of v and w is adjacent to both.

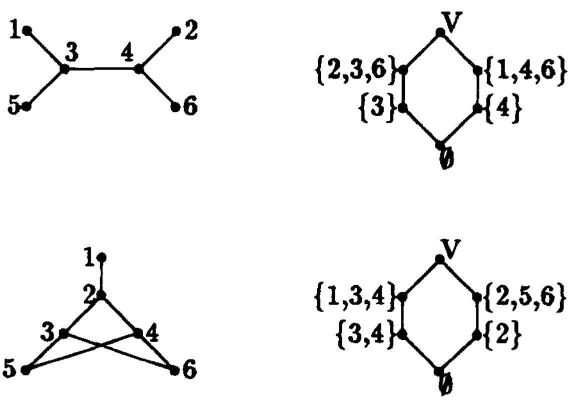

Fig. 1. Two graphs with the same corresponding orthocomplemented lattice.

The equivalence classes $[v]$ of the vertex set V are completely disconnected graphs and if v and w are adjacent, then v is adjacent to every vertex of the T-class $[w]$. So we can define the graph G/T with the T-classes as vertices and two T-classes $[v]$ and $[w]$ are adjacent iff v and w are adjacent in G.

It is easy to see that the following theorem holds for the two graphs G and G/T:

Theorem 3. The orthocomplemented lattices $H(G)$ and $H(G/T)$ are isomorphic.

Theorem 4. Let $G = (V, E)$ be a connected graph, different from K_2, with the following property: The only nontrivial H-closed subsets of V are the vertices of degree ≥ 2 and their neighborhoods. Then

$$H_0(G) \cong K_2 \otimes \overline{G}$$

where $H_0(G)$ results from the Hasse diagram of $H(G)$ by deleting the points corresponding to $\emptyset$ and V, and $\overline{G}$ is obtained from G by deleting all pendant vertices.

The assumption of this theorem is served by all trees different from K_2, all circles C_n with $n \neq 4$, and all cactuses different from K_2 and without any block which is a circuit of length 4. This means for two graphs G_1 and G_2 of the upper structure the orthocomplemented lattices $H(G_1)$ and $H(G_2)$ are isomorphic iff $\overline{G}_1$ and $\overline{G}_2$ are isomorphic, which is the second result of Dörfler.

3. NEW RESULTS

In the following the set of all nontrivial H-closed subsets of V is denoted by $H_0(G)$:

$$H_0(G) = H(G)\backslash\{\emptyset, V\}$$

Let now A be such a nontrivial H-closed set: $A \in H_0(G)$. By adding a vertex $w \notin V$ we construct the following graph G_A:

$$G_A = (V_A, E_A) = (V \cup \{w\}, E \cup \{[v, w]: v \in A\})$$

For the nontrivial H-closed sets of G_A the following theorem holds:

Theorem 5. A subset $C \subset V_A$ is related to G_A a H-closed set iff

$$C \in \{B: B \in H_0(G) \wedge R_G(A) \not\subseteq B\}$$

or

$$C \in \{B \cup \{w\}: B \in H_0(G) \wedge R_G(A) \subseteq B\}$$

Proof. To prove this theorem we have to divide $H_0(G)$ into two sets

$$H_1 = \{B: B \in H_0(G) \wedge R_G(A) \not\subseteq B\}$$

$$H_2 = \{B: B \in H_0(G) \wedge R_G(A) \subseteq B\}$$

Then we show that for $B \in H_1$, B is H-closed in G_A and for $B \in H_2$, $B \cup \{w\}$ is H-closed in G_A.

The second step is to show that $C \setminus \{w\}$ is an element of H_1 if $w \notin C$ and an element of H_2 otherwise.

This proof is very easy but too long to write down here. ∎

With the knowledge of Theorem 5 we can assign to each H-closed subset of V an H-closed subset of V_A as follows:

$$f(B) = \begin{cases} B & \text{if} \quad R_G(A) \not\subseteq B \\ B \cup \{w\} & \text{if} \quad R_G(A) \subseteq B \end{cases}$$

This is compatible with the partial order and the orthocomplementation.

With these results we get the following:

Theorem 6. The orthocomplemented lattices belonging to G and G_A are isomorphic.

The two results of Dörfler I mentioned above are only special cases of this theorem.

I suppose that we can reverse the last theorem. That is, we can reduce two graphs with the same orthocomplemented lattice by a construction corresponding to Theorem 6 to the same graph.

4. A CONSTRUCTION TO REDUCE GRAPHS

Because of this supposition I tried to find an algorithm to reduce a graph $G = (V, E)$ to the minimal graph. Such a reduction must satisfy the following conditions:

$G = (V, E)$ is reduced to $\overline{G} = (\overline{V}, \overline{E})$ with

$$\overline{V} = V \setminus \{v\}$$

$$\overline{E} = E \setminus \{[v_1, v_2]: v_1 = v \text{ or } v_2 = v\}$$

such that:

1. $R(\{v\})$ is H-closed in $\overline{G}$, too.
2. $R(\{v\}) \subset V \setminus \{v\}$ holds.

Such a construction is the following:

First we look for maximal bipartite subgraphs with components I_1 and I_2. Now we can delete a vertex v from a component, for example, I_1, iff:

1. $I_1 \backslash \{v\} \neq \emptyset$.
2. $R(\{v\}) = I_2$.
3. $R(I_1 \backslash \{v\}) = I_2$.

Example. As an example I will reduce the graph of Fig. 2. A maximal bipartite subgraph is the graph with the components

$$I_1 = \{1, 5\}$$

$$I_2 = \{2, 3, 4\}$$

and the edges

$$[1, 2], [1, 3], [1, 4], [5, 2], [5, 3], [5, 4]$$

It is easy to see that the vertex $v = 5$ satisfies the above three conditions. So we can delete this vertex in our first step. The graph remaining consists of four vertices.

Now we repeat the procedure with this graph. We find here a maximal complete bipartite subgraph with the components

$$I_1 = \{2, 4\}$$

$$I_2 = \{1, 3\}$$

and the corresponding edges. The vertices of the second component do not satisfy the second condition, but both vertices of the first component satisfy all three conditions. So we can delete one of them, for example, $v = 4$. Then we get the graph K_3.

This graph cannot be reduced. This means it is our minimal graph. Actually we can see that the corresponding orthocomplemented lattice is the same as above.

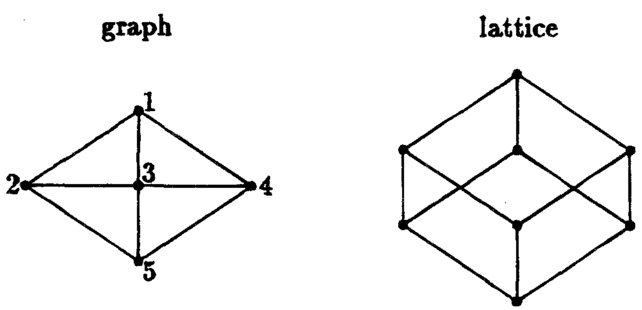

Fig. 2. Example.

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